Stochastic Strategies for Optimal Investment in a Defined Contribution (DC) Pension Fund

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Abstract
In this paper, optimal investment strategy for Defined Contribution (DC) pension fund with multiple contributors with stochastic rate of contribution is investigated. An optimized problem is derived using Hamilton Jacobi equation thereby applying Legendre transformation and duality theory in solving the optimal investment strategy for both CRRA and CARA utility function. Hence, we obtained an explicit solution for CARA utility function thereby extending the result in Dawei and Jingyi (2014).

Keywords: *CRRA, CARA, DC, Pension fund, Optimal Strategies, Legendre Transform, constant rate, stochastic rate*

1. Introduction

The need to understudy investment strategies has become one of the growing trends in the financial institutions. It has over the years play vital role in studying financial dynamics especially now that every institution is looking forward to maximize profit while reducing risk. The optimal investment strategy help financial institutions such as banks, insurance companies, pension managers etc. to determine the best possible way in which its wealth can be invested between risk free asset (cash and bond) and risky asset (stocks) with the aim of making interest to take care of future liabilities and subsequent investment. This strategy is not constant but varies with time and requires constant evaluation to determine at each time the best way to invest and maximize profit while reducing the risk.

Defined contribution pension is very crucial in retirement income system in a lot of countries and there is a growing trend to automatically involve all workers in it. In as much the DC scheme is relatively new compared to the defined benefit (DB) pension scheme, it forms a determining factor of the old age income adequacy for future retirees. This system underscores the need to understand better the risks that affect the income provided by this plan.. The most commonly used utility functions are the constant relative risk aversion (CRRA) see Cairns et al.(2006), Gao (2008), Boulier et al. (2001), Deelstra et al (2003), Xiao et al (2007) and constant absolute risk aversion (CARA) Battocchio and Menoncin (2004), Gao (2009).

There already existence of numerous literatures on optimal asset allocation for pension funds. Such as, Gao (2008) who studied an asset allocation problem under a stochastic interest rate. Boulier et. al (2001) studied optimal investment for DC with stochastic interest rate and Battocchio and Menoncin (2004)] where the interest rate was Vasicek model, Chubing and Ximing (2013), Deelstra et al (2003) and Gao (2008), studied the affine interest rate which include the Cox- Ingeroll- Ross (CIR) model and Vasicek model. Recently, more attention has been given to constant elasticity of Variance (CEV) model in DC pension fund investment strategies. As Geometric Brownian motion (GBM) can be considered as a special case of the (CEV) model, such work extended the research of Xio et al (2007) where they applied (CEV) model to derive dual solution of a CRRA utility function via Legendre transform, also Gao (2009) extended the work of Xiao et al (2007) by obtaining solutions for investor with CRRA and CARA utility function. Blake etal,(2012) investigate an asset allocation problem under a loss-averse preference. Cairns et.al.(2006) consider a stochastic salary income of a pension beneficiary and find the investment strategy which maximizes the expected power utility of the ratio of the terminal fund and the terminal salary. Korn et al. (2011), investigate a utility optimization problem for a DC pension plan with a stochastic salary income and a stochastic contribution process in a regime-switching economy. Their main interest lies in solving a filtering problem since they assume that the states of the economy are modelled by a hidden Markov chain. recently Dawei and Jingyi (2014) extended the work of Gao (2009) by modelling pension fund with multiple contributors where benefit payment are made after retirement , he went on to find the explicit solution for CRRA and CARA using power transformation method. In this paper we extend the model of Dawei and Jingyi (2014) from that of constant rate of contribution to that with stochastic rate and obtain an optimized problem using the Hamilton Jacobi equation. We then applied the Legendre transformation method and dual theory to solve the optimized problem for the optimal investment strategy for both CRRA and CARA utility function and compare the solution with that of Dawei and Jingyi (2014)

2. Mathematical Model

Financial Market

We assume that the market is made up of risk free asset (cash and bond) and risky asset (stock). Let (Ω, F, P) be a complete probability space where Ω is a real space and P is a probability measure, $\{W_s(t), W_t(t)\}$ is a standard two dimensional motion such that they orthogonal to each other. F is the filtration and denotes the information generated by the Brownian motion $\{W_S(t), W_t(t)\}$.

Risk Free Asset

Let $B(t)$ denote the price of the risk free asset, it model is given as

$$dB(t) = rB(t)dt$$

$$\frac{dB(t)}{B(t)} = rdt, \qquad (1)$$

Risky Asset

Let $S_t(t)$ denote the risky asset and its dynamics is given based on its stochastic nature and the price process described by the CEV model in Gao (2009) as

$$dS(t) = \mu S dt + KS^{\beta+1} dW_t(t)$$

$$\frac{dS(t)}{S(t)} = \mu dt + KS^{\beta} dW_t. \qquad (2)$$

Whereμ an expected instantaneous rate of return of the risky asset and satisfies the general condition $\mu > r_0$. $K{S_t}^{\beta}$ is the instantaneous volatility, and β is the elasticity parameter and satisfies the general condition $\beta < 0$.

In DC pension fund system with multiple contributors, we assume the followings

(1) Payment are made only to those who have retired
(2) Payment continues till the death of plan contributors
(3) Death contributors are automatically deleted from the system

From the above assumptions, the payment is a stochastic process. We assume the Brownian motion with drift as follows

$$dC(t) = adt - bdW_s(t) \qquad (3)$$

where a and b are positive constants and denote the amount given to the retired contributors and that which is due death contributors which are out of the system

We consider that in a DC fund system, contributors have the obligation to remit a specific percentage of their income to the pension account at the end of each month; also the contributors have the liberty to contribute an additional quota of their income to the pension account. Based on this, we consider a stochastic rate of contribution. We assume that the number of contributors is constant and the contribution rate is modelled as follows

$$dY = cdt + KS^{\beta}dW_t(t) \tag{4}$$

Where $c = (1+\theta)a$ with safty loading $\theta > 0$

$c = (1+\theta)a$, with safety loading $\theta > 0$. If there is no investment, the dynamics of the surplus is given by

$$dR(t) = dY - dC(t) = \theta adt + KS^{\beta}dW_t(t) + bdW_s(t) \tag{5}$$

Pension Wealth

Let $X(t)$ denote the wealth of pension fund at $t \in [0, T]$, let π denote the proportion of the pension fund invested in the risky asset S_t and $1-\pi$, the proportion invested in risk free asset . hence the dynamics of the pension wealth is given by

$$dX(t) = \pi X(t)\frac{dS(t)}{S(t)} + (1-\pi)X(t)\frac{dB(t)}{B(t)} + dR(t) \tag{6}$$

Substituting (1) and (2) into (5) we have

$$dX(t) = \left[\left(\pi X(t)(\mu - r)\right) + rX(t) + \theta a\right]dt + (\pi X(t)+1)KS^{\beta}dW_t(t) + bdW_s(t) \tag{7}$$

3. Optimization Problem

In this section we are interested in maximizing the utility of the plan contributors' terminal relative wealth. Assume we represent π as the strategy and we define the utility attained by the contributors from a given state x at time t as

$$H_{\pi}(t,s,x) = E_{\pi}\left[U\left(X(T)\right) \mid S(t) = s, X(t) = x\right], \tag{8}$$

Where t is the time, S is the price of the risky asset and x is the wealth. Our interest here is to find the optimal value function

$$H(t,s,x) = \sup_{\pi} H_{\pi}(t,s,x) \tag{9}$$

and the optimal strategy π such that

$$H_{\pi^*}(t,s,x) = H(t,s,x). \tag{10}$$

The Jacobi Hamilton-Jacobi-Bellman (HJB) equation associated with the optimization problem is

$$H_t + \mu s H_s + (rx + \theta a)H_x + \frac{1}{2}k^2 s^{2\beta+2}H_{ss} + \frac{1}{2}b^2 H_{xx} + \sup\left\{\frac{1}{2}(\pi x+1)^2 k^2 s^{2\beta} H_{xx} + \pi x(\mu - r)H_x + (\pi x + 1)k^2 s^{2\beta+1}H_{xs}\right\} = 0. \tag{11}$$

Differentiating equation (11) with respect to π, we obtain the first order maximizing condition as

$$x(\pi x + 1)k^2 s^{2\beta}H_{xx} + xk^2 s^{2\beta+1}H_{xs} + x(\mu - r)H_x = 0 \tag{12}$$

Solving equation (12) for π we have

$$\pi^* = -\frac{\left[(\mu - r)H_x + k^2 s^{2\beta+1}H_{xs} + k^2 s^{2\beta}H_{xx}\right]}{xk^2 s^{2\beta}H_{xx}}. \tag{13}$$

Substituting (13) into (11), we have

$$H_t + \mu s H_s + (rx + \theta a)H_x + \frac{1}{2}k^2 s^{2\beta+2}H_{ss} + \frac{1}{2}b^2 H_{xx} +$$
$$\sup\left\{\frac{1}{2}\left(-\frac{\left[(\mu - r)H_x + k^2 s^{2\beta+1}H_{xs} + k^2 s^{2\beta}H_{xx}\right]}{xk^2 s^{2\beta}H_{xx}}\right)x + 1)^2 k^2 s^{2\beta}H_{xx} -\right.$$

$\frac{[(\mu-r)H_x+k^2s^{2\beta+1}H_{xs}+k^2s^{2\beta}H_{xx}]}{xk^2s^{2\beta}H_{xx}}x(\mu-r)H_x+(-\frac{[(\mu-r)H_x+k^2s^{2\beta+1}H_{xs}+k^2s^{2\beta}H_{xx}]}{xk^2s^{2\beta}H_{xx}}x+1)k^2s^{2\beta+1}H_{xs}\Big\}=0..$ (14)

So that

$H_t+\mu sH_s+(rx+\theta a)H_x+\frac{1}{2}k^2s^{2\beta+2}\left[H_{ss}-\frac{H_{xs}{}^2}{H_{xx}}\right]+\frac{1}{2}b^2H_{xx}-\frac{(\mu-r)^2}{2k^2s^{2\beta}}\frac{H_x{}^2}{H_{xx}}-(\mu-r)s\frac{H_xH_{xs}}{H_{xx}}-(\mu-r)H_x=0.$ (15)

$H_t+\mu sH_s+(rx+\theta a-\mu+r)H_x+\frac{1}{2}k^2s^{2\beta+2}\left[H_{ss}-\frac{H_{xs}{}^2}{H_{xx}}\right]+\frac{1}{2}b^2H_{xx}-\frac{(\mu-r)^2}{2k^2s^{2\beta}}\frac{H_x{}^2}{H_{xx}}-(\mu-r)s\frac{H_xH_{xs}}{H_{xx}}=0$ (16)

4. Legendre Transformation

Considering the fact that the differential equation in (16) is a nonlinear partial differential equation and cumbersome to solve, we will apply the Legendre transform and dual theory to transform it to a linear partial differential equation

Theorem 4.1 : Let $f: R^n \to R$ be a convex function for $z > 0$, define the Legendre transform
$L(z) = \max_x\{f(x) - zx\}$, (17)
where $L(z)$ is the Legendre dual of $f(x)$. Jonsson and Sircar (2002)
Since $f(x)$ is convex, from theorem 4.1 we defined the Legendre transform
$\widehat{H}(t,s,z) = \sup\{\, H(t,s,x) - zx \mid 0 < x < \infty\,\}\ 0 < t < T.$ (18)
where $\widehat{H}$is the dual of H and $z > 0$ is the dual variable of x.
The value of v where this optimum is attained is denoted by $g(t,s,z)$, so that
$g(t,s,z) = \inf\{\, x \mid H(t,s,x) \geq zx + \widehat{H}(t,s,z)\,\}\ 0 < t < T.$ (19)

The function g and $\widehat{H}$ are closely related and can be refers to as the dual of H. These functions are related as follows
$\widehat{H}(t,s,z) = H(t,s,g) - zg.$ (20)
Where
$g(t,s,z) = x, H_x = z,\ \ g = -\widehat{H}_{z.}$

At terminal time, we denote
$\widehat{U}(z) = \sup\{\, U(x) - zx \mid 0 < x < \infty\,\},$
and
$G(z) = \sup\{x \mid U(v) \geq zx + \widehat{U}(z).$
As a result
$G(z) = (U')^{-1}(z),$ (21)
where G is the inverse of the marginal utility U and note that $H(T,s,x) = U(x)$
At terminal time T, we can define
$g(T,s,z) = \inf_{x>0}\{\, x \mid U(x) \geq zx + \widehat{H}(t,s,z)\,\}\, and\ \widehat{H}(t,s,z) = \sup_{x>0}\{\, U(x) - zx\}$
so that
$g(T,s,z) = (U')^{-1}(z).$ (22)
Next we differentiate (20) with respect to $t, s, and\ x$

$H_t = \widehat{H}_t,\ \ H_s = \widehat{H}_s\,, H_x = z, H_{sx} = \frac{-\widehat{H}_{sz}}{\widehat{H}_{zz}}, H_{xx} = \frac{-1}{\widehat{H}_{zz}}\ ,\ \ H_{sx} = \widehat{H}_{ss} - \frac{\widehat{H}_{sz}{}^2}{\widehat{H}_{zz}}.$ (23)

Substituting (23) into (16), we have

$$\widehat{H}_t + \mu s\widehat{H}_s + (rx + \theta a - \mu + r)z + \frac{1}{2}k^2 s^{2\beta+2}\widehat{H}_{ss} - \frac{1}{2}b^2\frac{1}{\widehat{H}_{zz}} - \frac{z^2(\mu-r)^2}{2k^2s^{2\beta}}\widehat{H}_{zz}{}^2 - (\mu - r)sz\widehat{H}_{zz} = 0 \quad (24)$$

and

$$\pi^* = \frac{[(\mu-r)z\widehat{H}_{zz} - k^2s^{2\beta+1}\widehat{H}_{sz} - k^2s^{2\beta}].}{xk^2s^{2\beta}} \quad (25)$$

Differentiating (24) and (25) with respect to z and using $x = g = -\widehat{H}_z$, we have

$$g_t + rsg_s - rg + (\mu - \theta a - r) + \frac{1}{2}k^2s^{2\beta+2}g_{ss} + \left(\frac{(\mu-r)^2}{k^2s^{2\beta}} - r\right)zg_z + \frac{1}{2}b^2\frac{g_{zz}}{g_z{}^2} + \frac{z^2(\mu-r)^2 g_{zz}}{2k^2s^{2\beta}} - (\mu - r)szg_{sz} = 0 \quad (26)$$

and

$$\pi^* = -\frac{[(\mu-r)zg_z - k^2s^{2\beta+1}g_s + k^2s^{2\beta}]}{gk^2s^{2\beta}} \quad (27)$$

5. CRRA and CARA Utility Function

In this section we attempt to find the explicit solution for the CRRA and CARA utility functions.

CRRA Utility Function

Assume the investor takes a power utility function

$$U(x) = \frac{x^p}{p}, \quad p < 1,\ p \neq 0 \quad (28)$$

The relative risk aversion of an investor with utility described in (28) is constant and (28) is a CRRA utility.

We assume a solution to (26) with the following form

$$g(t,s,z) = v(t,s)\left[z^{\frac{1}{p-1}}\right] + y(t), \qquad y(T) = 0,\ v(T,s) = 1.$$

Then

$$g_t = v_t z^{\frac{1}{p-1}} + y',\ g_z = -\frac{v}{1-p}z^{\left(\frac{1}{p-1}-1\right)},\ g_{sz} = -\frac{v_s}{1-p}z^{\left(\frac{1}{p-1}-1\right)},$$

$$g_{zz} = \frac{(2-p)v}{(1-p)^2}z^{\left(\frac{1}{p-1}-1\right)}, g_s = v_s z^{\frac{1}{p-1}}, g_{ss} = h_{ss}z^{\frac{1}{p-1}}. \quad (29)$$

Substitute (29) into (26) we have

$$\left[v_t + rsv_s - rv + \frac{1}{2}k^2s^{2\beta+2}v_{ss} - \frac{v}{1-p}\left(\frac{(\mu-r)^2}{k^2s^{2\beta}} - r\right) + \frac{(\mu-r)^2}{2k^2s^{2\beta}}\frac{(2-p)v}{(1-p)^2} + (\mu - r)s\frac{v_s}{1-p}\right]z^{\frac{1}{p-1}} + \left(\frac{1}{2}b^2\frac{v}{2-p}\right)z^{\frac{-1}{p-1}} + (y' + yr - \theta a - r + \mu) = 0. \quad (30)$$

From (30) we have

$$v_t + rsv_s - rv + \frac{1}{2}k^2s^{2\beta+2}v_{ss} - \frac{v}{1-p}\left(\frac{(\mu-r)^2}{k^2s^{2\beta}} - r\right) + \frac{(\mu-r)^2}{2k^2s^{2\beta}}\frac{(2-p)v}{(1-p)^2} - (\mu - r)s\frac{v_s}{1-p} = 0 \quad (31)$$

So that

$$\frac{1}{2}b^2\frac{v}{2-p} = 0 \quad (32)$$

And

$$y' + yr + (\mu - r - \theta a) = 0. \quad (33)$$

From equation (32), and since $b > 0$, then $v = 0$,
From (33) we have

$$w(t) = -\frac{(\mu-r-\theta a)}{r}(1 - e^{-r(t-T)}).$$

Observe that (31) reduces to

$$v_t + (\mu - rp)s\frac{v_s}{1-p} + \frac{1}{2}k^2s^{2\beta+2}v_{ss} + \frac{rp}{1-p}v + \frac{p(\mu-r)^2}{2k^2s^{2\beta}(1-p)^2}v = 0 \quad (34)$$

Solving (34) for v will give us a solution different from $v = 0$ as obtain from (32) which is a contradiction and hence we cannot find an explicit solution for the CRRA utility. This shows that the relative risk aversion of investor is constant.

CARA Utility Function

Assume the contributor takes an exponential utility

$$U(x) = -\frac{1}{q}e^{-qx}, \quad q > 0. \tag{35}$$

The absolute risk aversion of a decision maker with the utility described in (32) is constant and is a CARA utility

Since $g(T,s,z) = (U')^{-1}(z)$ with the CARA utility function we obtain

$$g(T,s,z) = -\frac{1}{q}lnz. \tag{36}$$

Hence we conjecture a solution to (26) with the following form

$$g(t,s,z) = -\frac{1}{q}[y(t)(lnz + v(t,s))] + u(t), \tag{37}$$

with boundary conditions $y(T) = 1,\ u(T) = 0, m(T,s) = 0$

$$g_t = -\frac{1}{q}[y'(t)(lnz + v(t,s)) + yv_t] + u'(t),$$

$$g_s = -\frac{1}{q}yv_s,\ g_z = -\frac{y}{qz}\ ,\ g_{zz} = \frac{y}{qz^2}, g_{ss} = -\frac{1}{q}yv_{ss},\ g_{sz} = 0. \tag{38}$$

Substituting (38) into (26), we have

$$[y'(t) - ry(t)]lnz + [-u'(t) + ru(t) + \mu - r + \theta a]q + [v_t + rsv_s + \frac{1}{2}k^2s^{2\beta+2}v_{ss} + \frac{(\mu-r)^2}{2k^2s^{2\beta}} - rv + \frac{y'}{y}v - r - \frac{1}{2}b^2]y = 0.$$

Such that

$$y'(t) - ry(t) = 0 \tag{39}$$

And

$$v_t + rsv_s + \frac{1}{2}k^2s^{2\beta+2}v_{ss} + \frac{(\mu-r)^2}{2k^2s^{2\beta}} - r - \frac{1}{2}b^2 = 0. \tag{40}$$

So that

$$-u'(t) + ru(t) - \mu + r + \theta a = 0 \tag{41}$$

Solving (39) and (41), we have

$$y(t) = e^{-r(t-T)} \tag{42}$$

and

$$u(t) = \frac{\mu-r-\theta a}{r}(1 - e^{-r(t-T)}). \tag{43}$$

Next we conjecture a solution of (40) with the following structure

$$v(t,s) = F(t) + G(t)s^{-2\beta},\ F(T) = 0, G(T) = 0 ,$$

$$v_t = F_t + G_ts^{-2\beta}, v_s = -2\beta Gs^{-2\beta-1}, v_{ss} = 2\beta(2\beta+1)Gs^{-2\beta-2}. \tag{44}$$

Substituting (44) into (40) we have

$$F_t + \beta(2\beta+1)k^2G - r - \frac{1}{2}b^2 + s^{-2\beta}\left[G_t - 2r\beta G + \frac{(\mu-r)^2}{2k^2}\right] = 0, \tag{45}$$

so that

$$F_t + \beta(2\beta+1)k^2G - r - \frac{1}{2}b^2 = 0, \tag{46}$$

and

$$G_t - 2r\beta G + \frac{(\mu-r)^2}{2k^2} = 0. \tag{47}$$

Solving (44) with the given condition gives;

$$G(t) = \frac{(\mu-r)^2}{4k^2r\beta}\left[1 - e^{2r\beta(t-T)}\right]. \tag{48}$$

Next substituting (48) into (46) and solving (46) with the given condition we have

$$F(t) = \left[\frac{(2\beta+1)(\mu-r)^2}{4r} - r - \frac{1}{2}b^2\right](T-t) - \left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1-e^{2r\beta(t-T)}\right)\right]. \quad (49)$$

$$v(t,s) = \left[\frac{(2\beta+1)(\mu-r)^2}{4r} - r - \frac{1}{2}b^2\right](T-t) - \left[\frac{(2\beta+1)(\mu-r)^2}{8r^2\beta}\left(1-e^{2r\beta(t-T)}\right)\right]$$
$$+\left[\frac{s^{-2\beta}(\mu-r)^2}{4k^2r\beta}\left(1-e^{2r\beta(t-T)}\right)\right]. \quad (50)$$

From (42), (43) and (50) we obtain an explicit solution for (26) for a CARA function given as in (37)
with

$$g_t = -\frac{1}{qz}e^{r(t-T)}. \quad (51)$$

$$g_s = \frac{1}{q}e^{r(t-T)}\frac{s^{-2\beta-1}(\mu-r)^2}{2k^2r}\left(1-e^{2r\beta(t-T)}\right)], \quad (52)$$

The optimal investment strategy is given as

$$\pi^* = -\frac{[(\mu-r)zg_z - k^2s^{2\beta+1}g_s + k^2s^{2\beta}]}{gk^2s^{2\beta}}$$

$$\pi^* = \frac{1}{q}\frac{(\mu-r)}{k^2s^{2\beta}g}e^{r(t-T)}\left[1+\frac{(\mu-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)\right] + \frac{1}{g} \quad (53)$$

If $ks^{\beta} = \sigma_*$, $R(\sigma_*) = \frac{(\mu-r)}{q{\sigma_*}^2}$, $Q(t) = 1 + \frac{(\mu-r)}{2r}\left(1-e^{2r\beta(t-T)}\right)$ $and\ g = x,$

then

$$\pi^* = g^{-1}(R(\sigma_*)\, Q(t) + 1)\,. \quad (54)$$

Let

$A = g^{-1}(R(\sigma_*)\, Q(t)$and $B = g^{-1}$, then (54) becomes

$$\pi^* = A + B \quad (55)$$

6. Discussion

Here we obtain a result different from that of Dawei and Jingyi (2014) when it was solved for constant rate of returns using power transformation method. We observed that with stochastic rate of contributions the optimal investment strategy for the risky asset is given by $A + B$which is different from that with constant rate of contribution as obtained in the work of Dawei and Jingyi (2014) which is just A. Thus our result extend the result in Dawei and Jingyi (2014) which implies that there will be disparity in the proportion to be invested in the risky asset in the case of stochastic rate of contribution and constant rate of contribution.

7. Conclusion

We studied optimal investment strategy for DC pension fund with multiple contributors with stochastic rate of contribution. We took into consideration the fact that members of the pension fund have the liberty of contributing an extra of their income to the pension scheme. We extended the model of Dawei and Jingyi (2014) from that of constant rate of contribution to that with stochastic rate and obtain an optimized problem using the Hamilton Jacobi equation. We then applied the Legendre transformation method and dual theory to solve the optimized problem for the optimal investment strategy for both CRRA and CARA utility function and obtained an explicit solution for CARA utility function different from that of Dawei and Jingyi (2014) but could not obtained an explicit solution for the CRRA function which was also the case in Dawei and Jingyi (2014) .We generalized the result of Dawei and Jingyi (2014) and observed that the proportion to be invested in the risky asset will be greater in the case of stochastic rate of contribution compared to that with constant rate of contribution.

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